Exploration of Toggle on Graphs

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Overview

1. Intro
2. About Math Research
   - How is Math Researched
   - Why is Math Researched
3. Toggle
   - Preliminary Info
   - What is Toggle
   - Results

Research Process

- Read Existing Literature

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- Experiment with associated objects

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- Attempt to prove

Research Process

- Read Existing Literature
- Experiment with associated objects
- Create Conjectures
- Attempt to prove
- Fail, try again

What is a proof

- As experiments are to most sciences, proofs are to mathematics

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- Rigorous argument for or against a statement

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What is a proof

- As experiments are to most sciences, proofs are to mathematics
- Rigorous argument for or against a statement
- An argument is a sequence of true statements that follow one another
- Make as few assumptions as possible

Why do we do Math Research

Why do we do Math Research

Because its fun

Why do we do Math Research

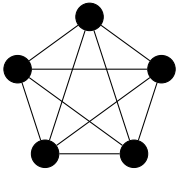
- Applications may exist in future

Why do we do Math Research

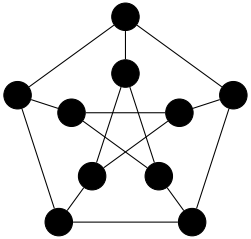
- Applications may exist in future
- Explore the proccess of creating precise arguments

Graphs

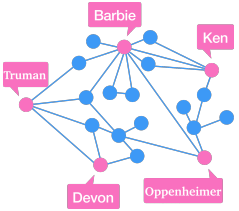
(a) Path Graph



(b) Complete Graph



(c) Petersen Graph



(d) Social Network

Games

Definition (Impartial Two-Player Game)

An **impartial two-player game** is a game where the allowable moves only depend on the board state or position and not on which of the two players is currently moving.

Definition (N-Position)

A two-player impartial game is in an **N-position** if that position secures a win for the **Next** player. A game is an **N-game** if the initial position is an N-position.

Definition (P-Position)

A two-player impartial game is in a **P-position** if that position secures a win for the **Previous** player. A game is a **P-game** if the initial position is a P-position.

Games in plain english

Definition (Impartial Two-Player Game)

- Two-Player Game
- Either player may make any move

Definition (N-Position)

- The next player to move has a winning strategy

Definition (P-Position)

- The previous player has a winning strategy

What is Toggle

- Impartial two player game played on a simple graph
- Players take turns turning off lights
- Last player to turn off a light wins

Heat Toggle

Valid Move

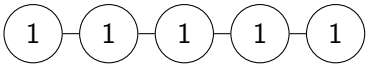
- Each light is initially on
- Toggling a light changes its flips its state and the state of its neighbors
- Only an on light may be toggled

Heat Toggle

Valid Move

- Each light is initially on
- Toggling a light changes its flips its state and the state of its neighbors
- Only an on light may be toggled
- More lights must be off after a move

Heat Toggle Example



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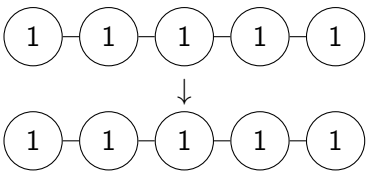

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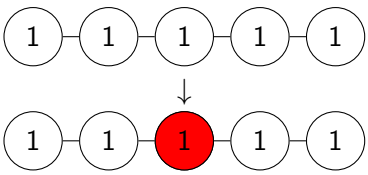

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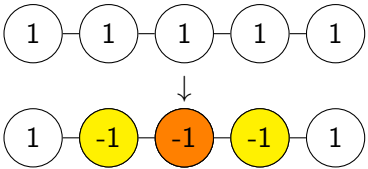

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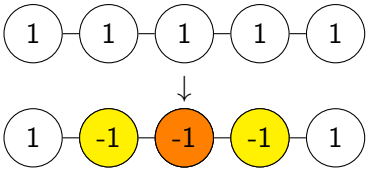

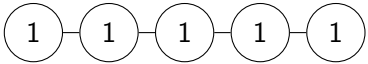

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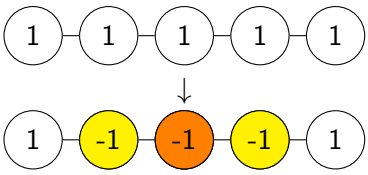

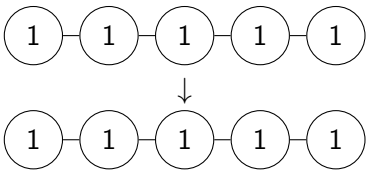

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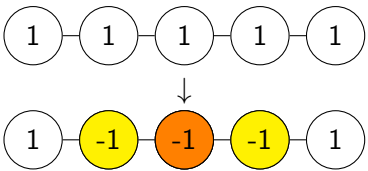

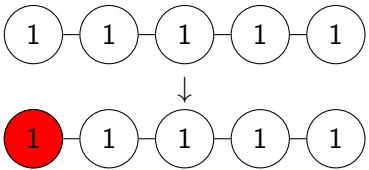

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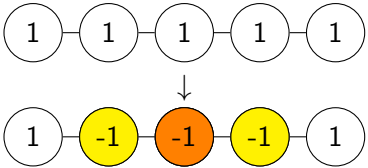

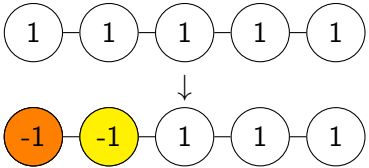

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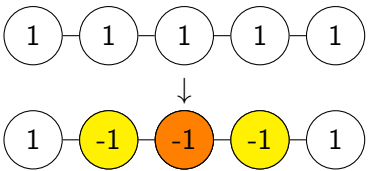

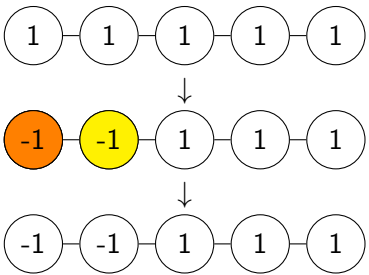

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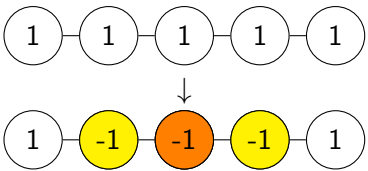

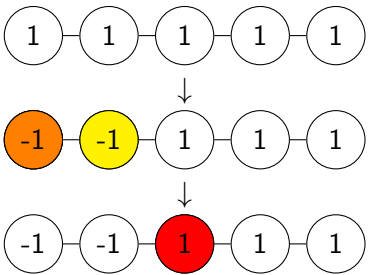

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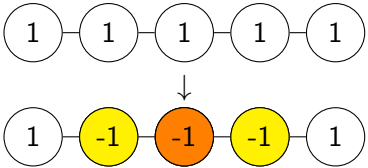

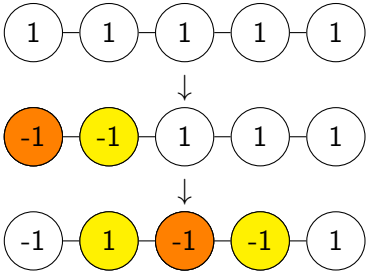

Sprague-Grundy

Theorem (Sprague-Grundy Theorem)

Every impartial game under normal play is equivalent to a Nim game played on a heap of a certain size.

Definition (Grundy Value)

The heap size of a Nim game is its **Grundy Value**. A **Grundy Value** of zero is a P-game while a nonzero **Grundy Value** is an N-game.

Lattice Graph

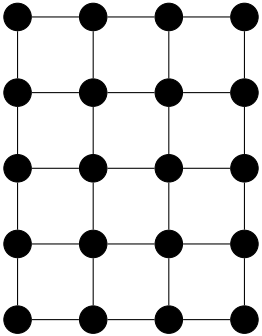

Figure 2: Example 4×5 Lattice Graph

Grundy Values of $n \times m$ Latticies

$n \setminus m$	1	2	3	4	5	6
1	1	1	1	2	2	0
2	1	1	2	0	3	1
3	1	2	1	1	3	0
4	2	0	1	0	1	0
5	2	3	3	1	2	0
6	0	1	0	0	0	...

Complexity Class

- How do we measure how difficult it is to compute the solution of a problem?

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  - Time
  - Memory

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- Why do we care about complexity classes?

Complexity Class

- How do we measure how difficult it is to compute the solution of a problem?
  - Time
  - Memory
- Why do we care about complexity classes?
  - If two problems have the same complexity class, we can convert one into the other
  - If the algorithim for one problem is easier than another, we can convert more difficult problem into a simpler one

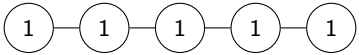
The Office

Figure 3: Scene from *The Office* Season 9 Episode 4

Charged Toggle

- After a light is toggled, it can not be toggled again
- The number of on lights does not need to decrease

Charged Toggle Example



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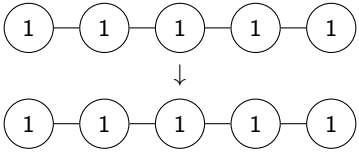

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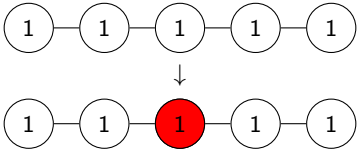

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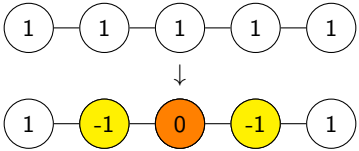

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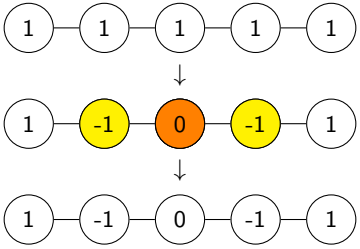

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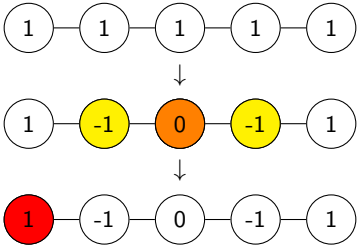

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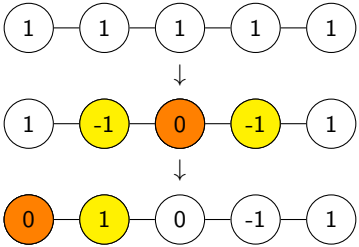

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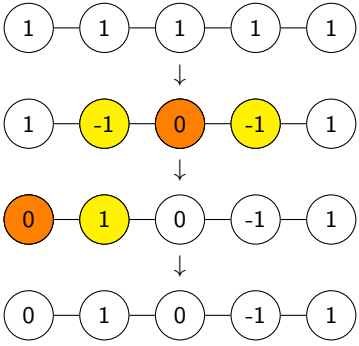

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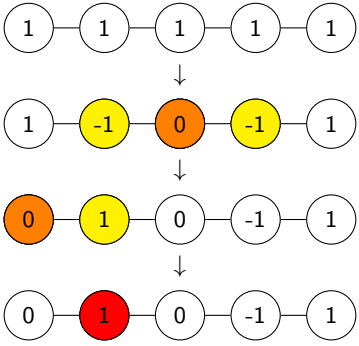

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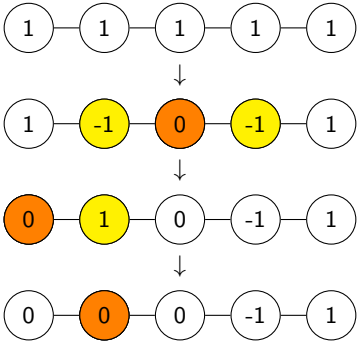

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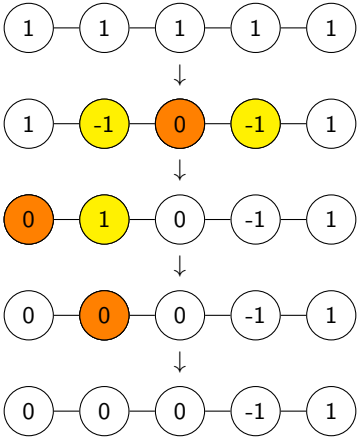

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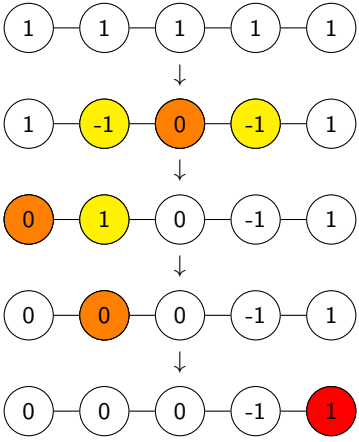

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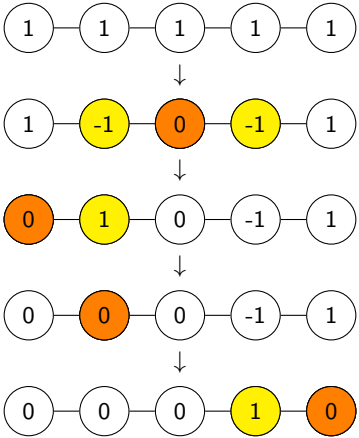

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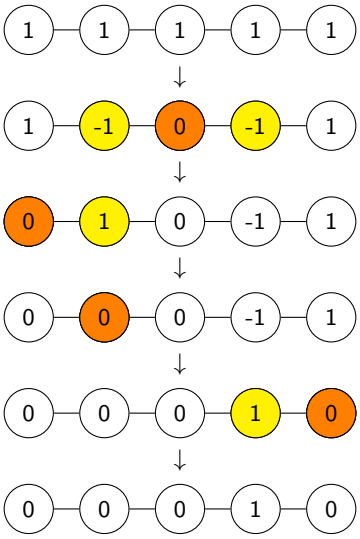

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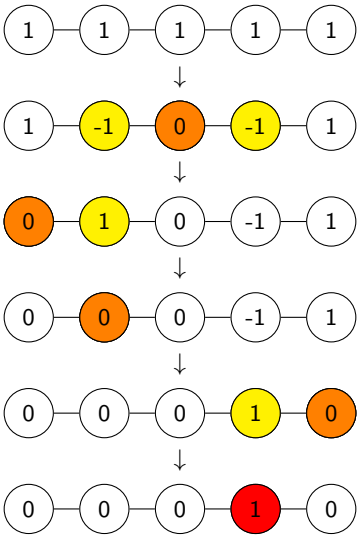

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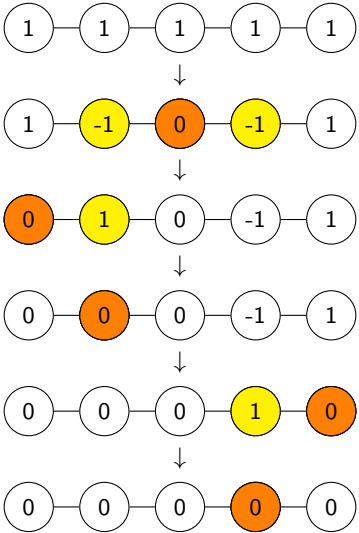

Charged Toggle can be modeled by Heat Toggle

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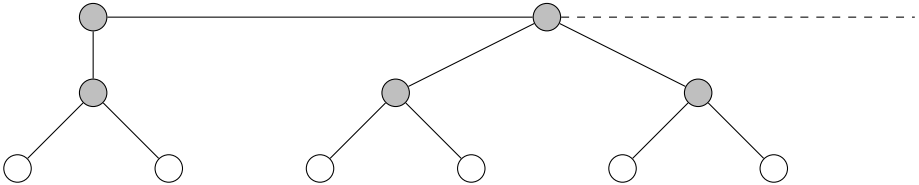

Heat Toggle is PSpace Complete

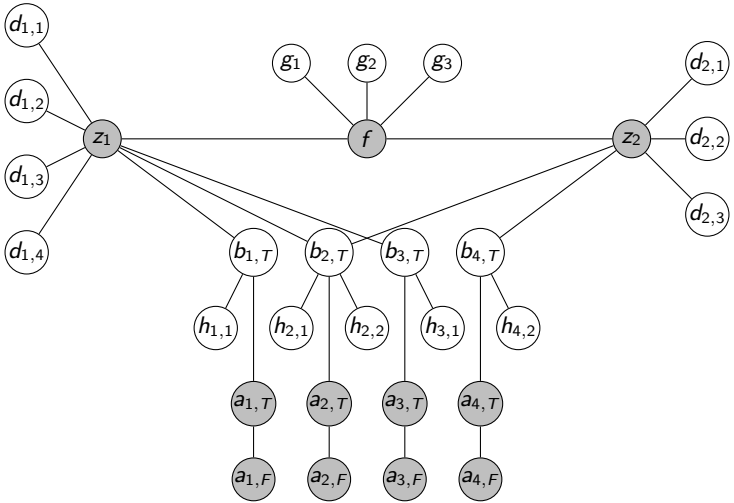

Figure 4: The expression $(x_1 \wedge x_2 \wedge x_3) \vee (x_2 \wedge x_4)$ as an Heat Toggle game

Acknowledgements

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MORAVIAN UNIVERSITY Diversity, Equity & Inclusion

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