Exploring Toggle Games on Graphs

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Impartial Games

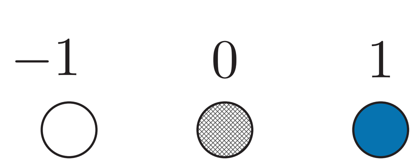
An **impartial two-player game** is a game where the allowable moves only depend on the board state or position and not on which of the two players is currently moving

A two-player impartial game is in an **N-position** if that position secures a win for the **Next** player. Conversely, a **P-position** is a state which secures a win for the **Previous** player. A game is an **N-game** or **P-game** if the initial position is an N-position or P-position respectively.

Theorem (Sprague-Grundy). *All impartial games can be analyzed by assigning a nonnegative integer value, often called the* ***Grundy value****, to each game position recursively. The Grundy value of a game is 0 if and only if the game is a P-game, i.e., the second player has a winning strategy regardless of the moves of the first player.*

Toggle games are impartial two-player games played on simple graphs with vertex weights from the set $\{-1, 0, 1\}$. Each vertex v has an initial weight $\omega(v) = 1$. Toggling a vertex negates its weight and the weights of its neighbors.

$\mathcal{G}(1^n)$ is the Grundy value for a Toggle game on a path graph P_n where every vertex v has a weight $\omega(v) = 1$.



Example Verticies and their Weights

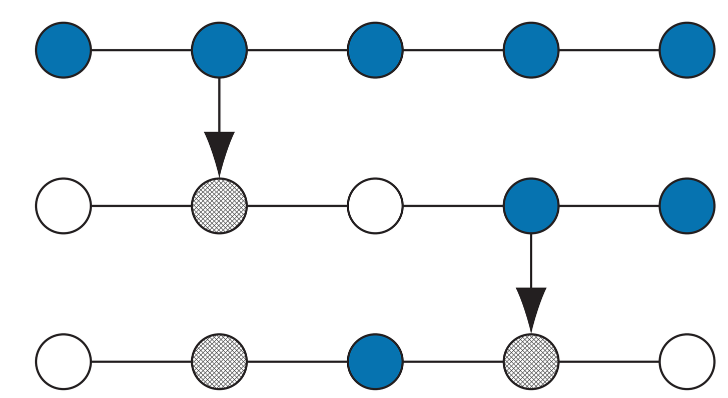
Heat Toggle

Every move must strictly decrease the sum of $\omega(v)$ over all $v \in V(G)$.

Heat Toggle Demo

Charge Toggle

In Charge Toggle, a move that toggles v sets $\omega(v) = 0$ instead of $\omega(v) = -1$.



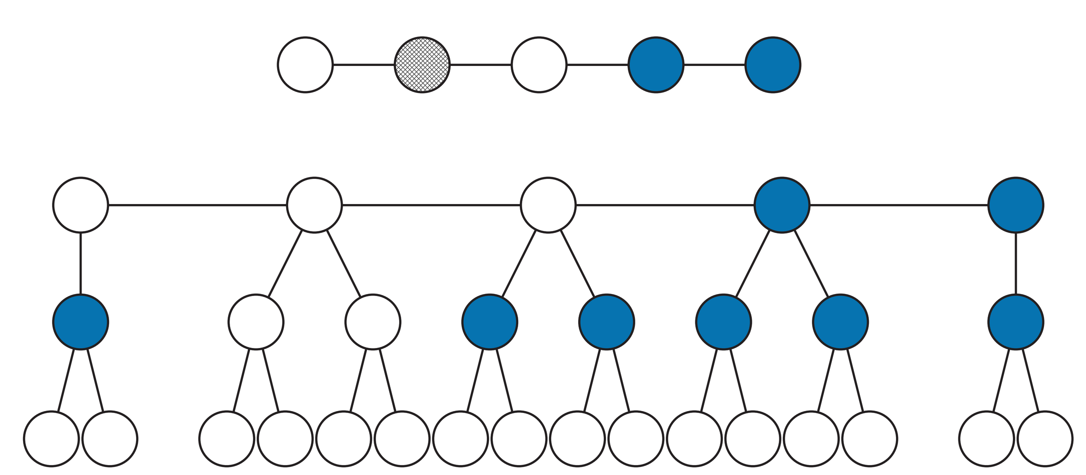
Results on Paths

Theorem. *In Heat or Charge Toggle on a path graph P_n, if $n \equiv 1 \pmod 2$ then the game is an N game.*

Theorem. *For $n \geq 0$, Charge Toggle on P_n is a P game if $n \equiv 0 \pmod 6$.*

Modeling Charge Toggle

Theorem. *Given a Charge Toggle game, there exists a Heat Toggle game with a bijection between their game states.*



Charge Toggle Game and it's Heat Toggle Model

Heat Charge Toggle

Heat Charge Toggle is a game where both the rules of Heat and Charge Toggle are enforced.

Conjecture. *For Heat Charge Toggle on a path graph with $n \geq 92$, the following equivalence holds*

$$\mathcal{G}(1^n) = \mathcal{G}(-1, 1^{n-1}, -1)$$

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Path Sinisterity

An **odious number** is one whose binary expansion contains an odd number of 1's. An **evil number**'s binary expansion contains an even number of 1's.

When playing Heat Charge Toggle on P_n

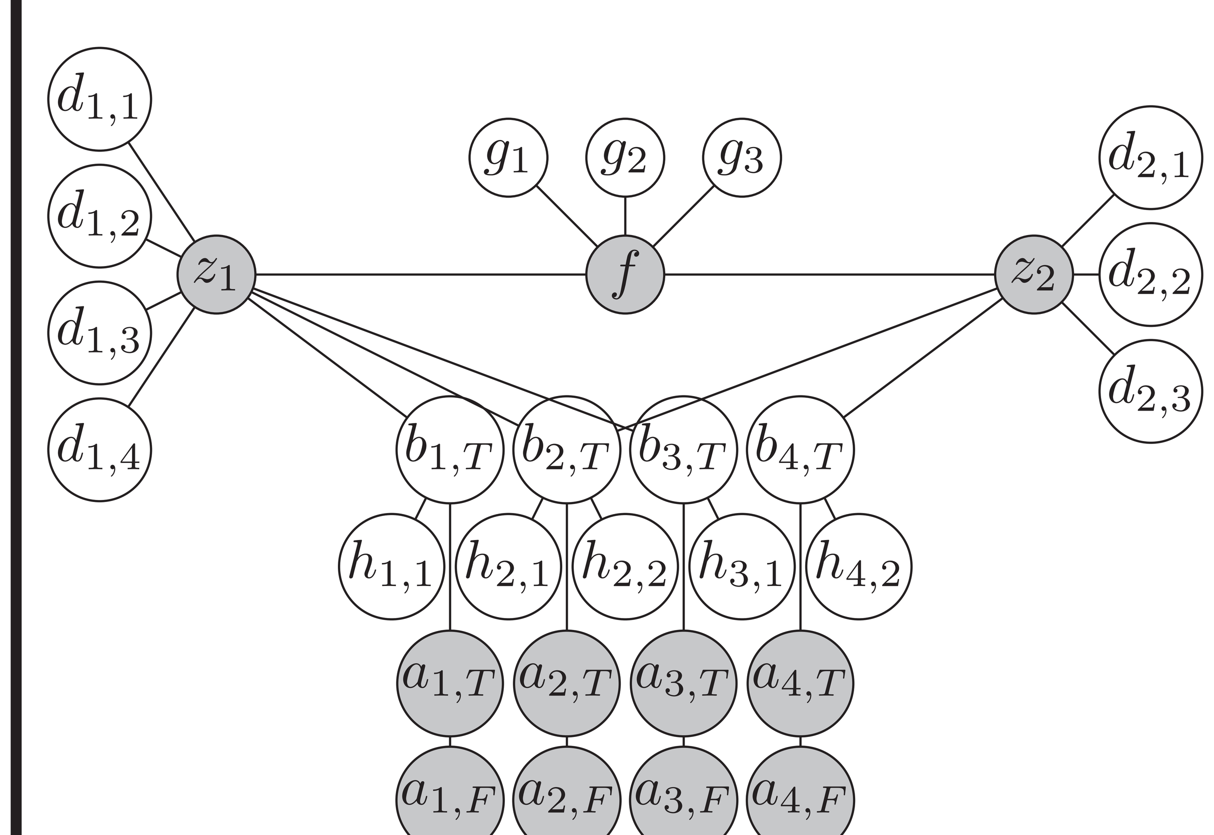
- If n is odd, $\mathcal{G}(-1, 1^n, -1)$ tends to be odious
- If n is even, $\mathcal{G}(-1, 1^n, -1)$ tends to be evil

The exceptions to the observation above are as follows:

n	$\mathcal{G}(-1, 1^n, -1)$		
1	0	89	5
4	2	103	5
6	1	105	6
9	3	108	4
14	2	124	7
22	2	129	6
27	3	141	3
30	4	171	10
35	5	258	4
41	3	84	16
58	7	407	18
59	3	458	16
72	4	11770548	32
84	4	25146268	32
87	6	27690032	32

Computational Complexity

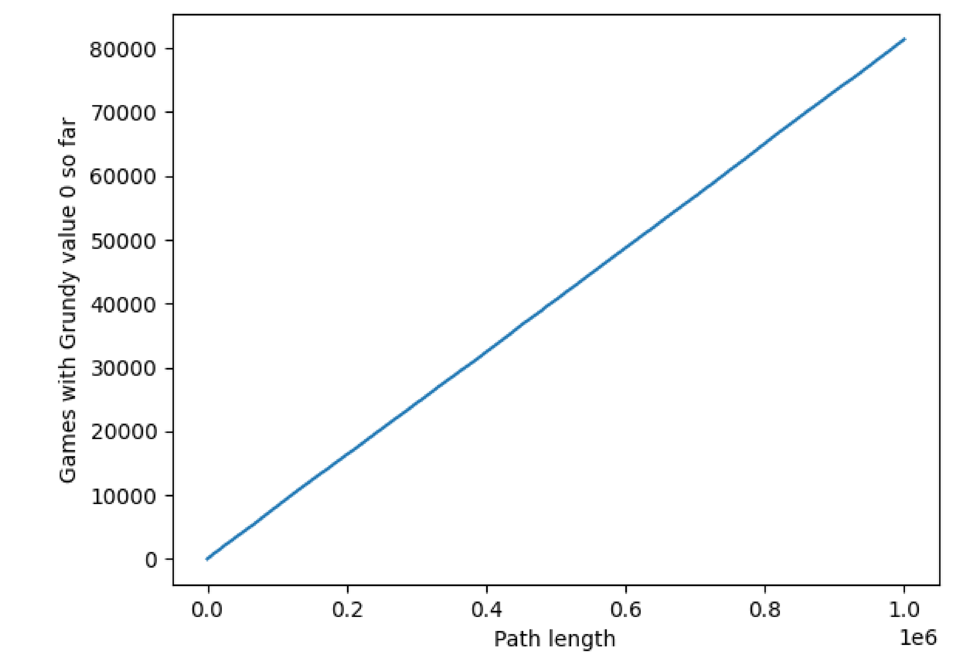
Theorem. *Heat Toggle is PSPACE-complete.*



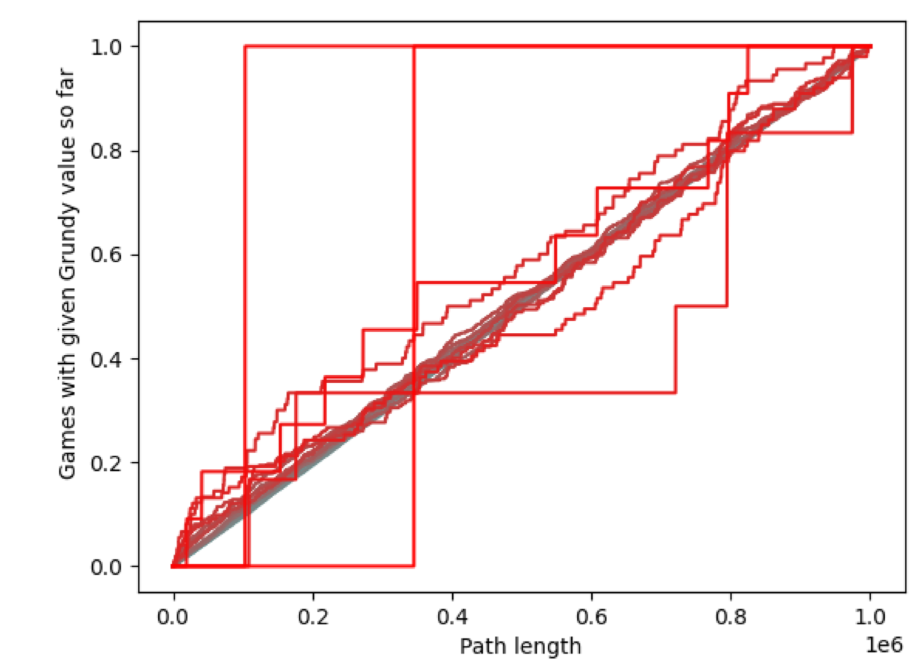
Example Heat Toggle game to solve boolean formula $(x_1 \wedge x_2 \wedge x_3) \vee (x_2 \wedge x_4)$

Grundy Value Distribution

For Heat Charge Toggle on paths with, the number of games with a given Grundy value grow as follows:



Occurence of Grundy value 0 as function of path length



Occurences of Grundy values as functions of path length

Future Work

- Prove or disprove the periodicity of the path sinisterity exceptions
- Create and Combine Heat Toggle Logic Gates

Acknowledgements

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