k-total bondage in Graphs

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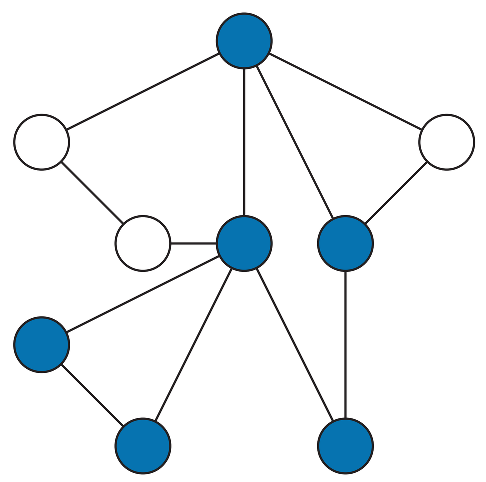
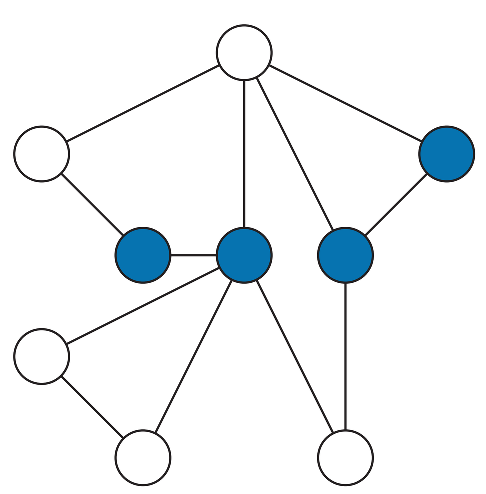
Total Domination

Let G be a connected simple graph with a vertex set $V(G)$ and edge set $E(G)$. A subset of vertices S is a **total dominating set** of G if every vertex in $V(G)$ is adjacent to some vertex from S.



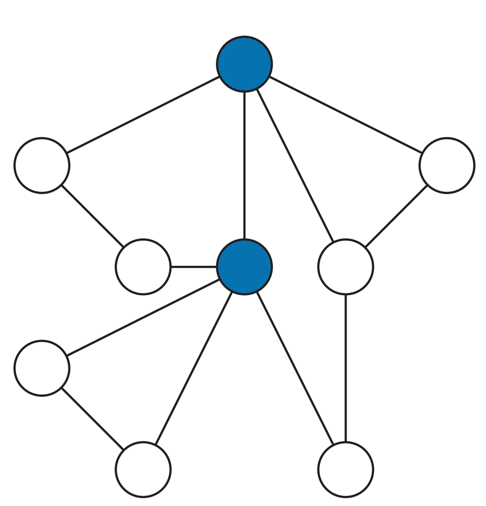
A graph with a total dominating set (blue)

A total dominating set S with no proper subsets that are total dominating sets is a **minimal total dominating set**.



A graph with a minimal total dominating set

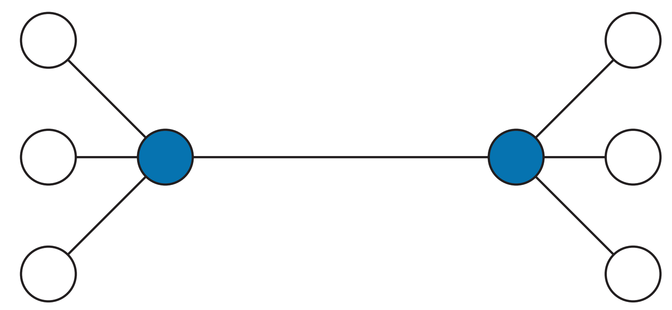
Let L be the set of all minimal total dominating sets of a graph G. A minimal total dominating S is a **minimum total dominating set** if there does not exist a set in L with a smaller cardinality. The cardinality of a minimum total dominating set is the **total domination number** of a graph. We notate this as $\gamma_t(G)$.



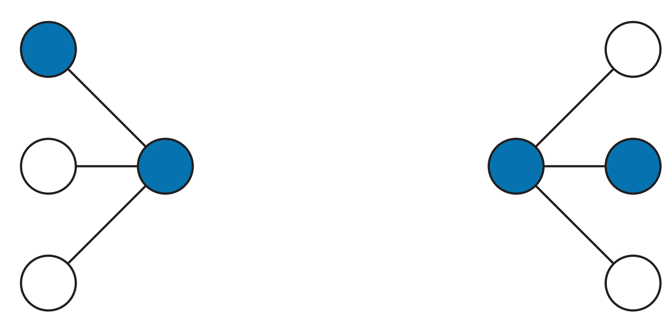
A graph G with a minimum total dominating set where $\gamma_t(G) = 2$

k-Total Bondage

The minimum number of edges to remove to increase the total dominating number of a graph by at least k is the **k-total bondage number** We notate the k-total bondage number of a graph G as $b_t^k(G)$.



A double star graph, $S_{3,3}$



Subgraph of $S_{3,3}$, showing that $b_t^2(S_{3,3}) = 1$

Bounds

For a connected graph G on n vertices with $n \geq 4$, the total domination number can increase by no more than

$$k \leq \begin{cases} n - \gamma_t(G) & n \equiv 0 \mod 2 \\ n - \gamma_t(G) - 1 & n \equiv 1 \mod 2 \end{cases}$$

Path Graphs

Henning and Yeo (2013) found for a path P_n where $n \geq 2$

$$\gamma_t(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1,3 \pmod 4 \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

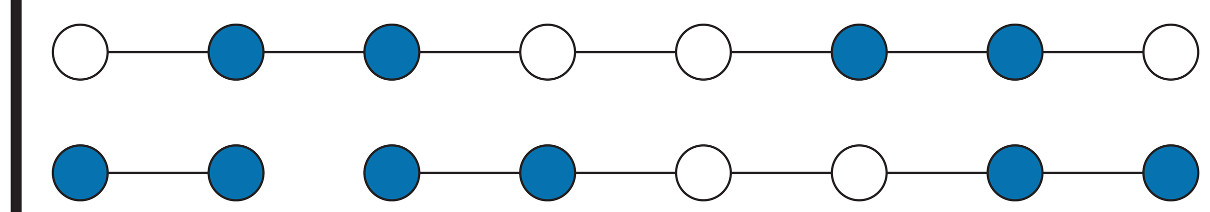
We found given two integers a and b where $a, b > 2$ the total domination number of a graph $P_a \cup P_b$ is no larger than the total domination number of the graph $P_{a+b-2} \cup P_2$.

$$\gamma_t(P_a \cup P_b) \leq \gamma_t(P_{a+b-2}) + \gamma_t(P_2)$$

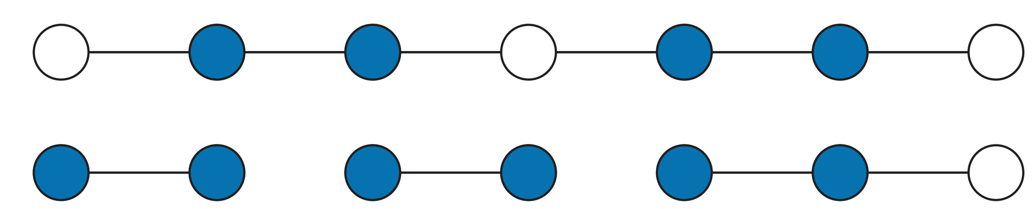
Path k-Total Bondage

The k-total bondage number of graph P_n for $k \leq 2\left\lfloor \frac{n}{2} \right\rfloor$ is

$$b_t^k(P_n) = \begin{cases} 2\left\lfloor \frac{k-1}{2} \right\rfloor + 1, & n \equiv 0 \pmod 4 \\ k, & n \equiv 1 \pmod 4 \\ 2\left\lfloor \frac{k-1}{2} \right\rfloor + 2, & n \equiv 2 \pmod 4 \\ k, & n \equiv 3 \pmod 4 \end{cases}$$



$\gamma_t(P_8) = 4 \qquad b_t^2(P_8) = 1$

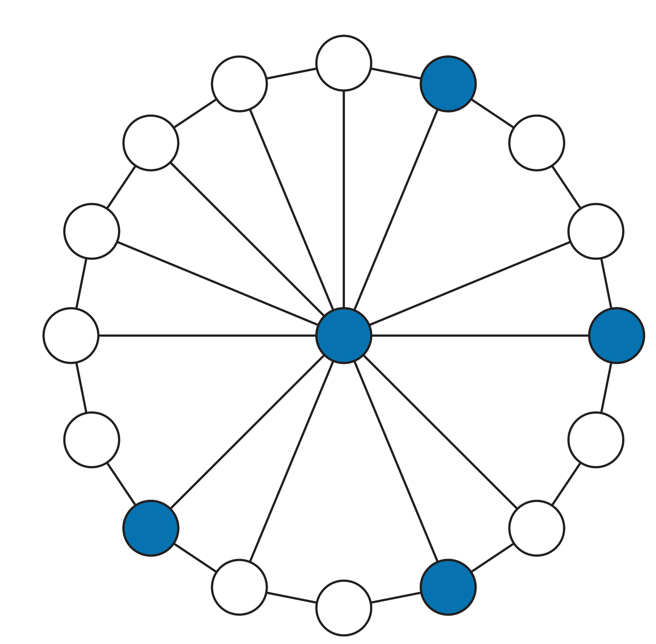


$\gamma_t(P_7) = 4 \qquad b_t^2(P_7) = 2$

Wheel Graphs

Let W_n be a wheel graph where $W_n = C_n + K_1$. For wheels W_n where $n \geq 3k$

$$b_t^k(W_n) = k + 1$$



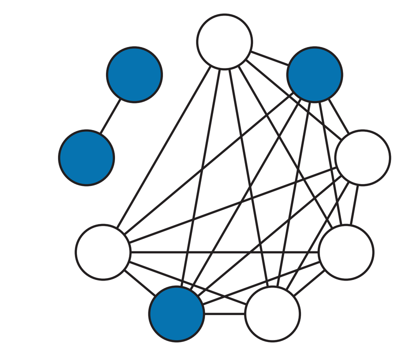
$b_t^3(W_{16}) = 4$

References

Henning, Michael A. and Anders Yeo (2013). *Total domination in graphs*. Springer Monographs in Mathematics. Springer, New York, pp. xiv+178.

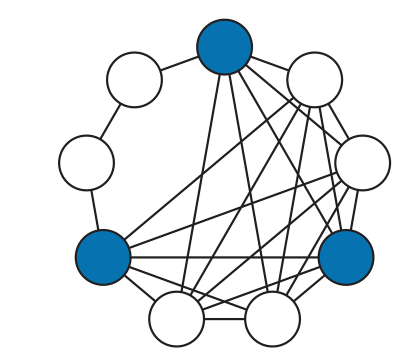
Complete Graphs

For a complete graph K_n, to increase the total domination by 2 the graph can be disconnected into $K_{n-2} \cup P_2$.



$b_t^2(K_9) = 14$

In order to increase the total domination number of K_n by 1 construct the graph $K_{n-2} \cup P_2$. Remove an edge between any two vertices in the K_{n-2} component then connect each vertex from the P_2 component to a vertex of a degree $n-3$.



$b_t^1(K_9) = 13$

In general, by succesively creating k P_2 components from a $K_n, K_{n-2}, \ldots, K_{n-2k+2}$ the total bondage number of a complete graph can be increased to any even number. Combining this strategy with the strategy to increase the total bondage number by 1 allows us to increase the total bondage number of a complete graph by any amount.

$$b_t^k(K_n) \leq \begin{cases} k(n - \frac{k+2}{2}) & k \equiv 0 \mod 2 \\ (k+1)(n - \frac{k+3}{2}) - 1 & k \equiv 1 \mod 2 \end{cases}$$

Future Work

We are working on a proving equality for the k-total bondage number of complete graphs. Additionally we have some work for bounds on complete bipartite graphs.

Acknowledgements

This research was funded by NSF Grant MPS-2150299 at the Moravian University REU.